1 Analytical Spherical Shadows

1.1 Assumptions

For the analytical soft shadows we make some assumptions before diving into the maths:

- All objects (sun, planets, moons) are spheres.
- The light intensity of the sun is at any point and from any direction constant (which isn't true obviously, see limb darkening).
- We ignore all other effects like atmospheric ones.

These assumptions result in a purely geometrical problem which is shown in figure 1.

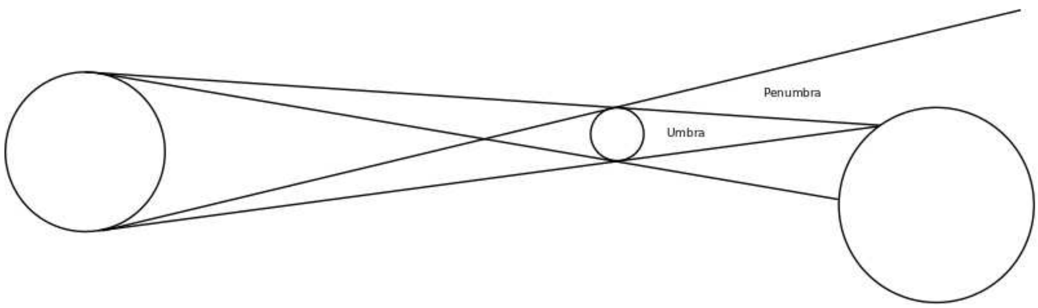

Figure 1: Geometrical problem shown from above. Left to right: Sun, moon, planet. The moon casts a shadow which can be split into Umbra and Penumbra.

The umbra is the area where the intensity of the direct illumination of the sun drops to zero. In the area of the penumbra the intensity is somewhere between 0 % and 100 % as the sun is only partially occluded by the moon.

1.2 The Problem

The problem we have is that we want to compute the relative intensity of the sun on a given point $\vec{P}$ on the surface of an object. The sun is given with the radius $R_\odot$ and the position $\vec{S}$ and the shadow casting object is also given having a radius r_M and a position $\vec{M}$.

1.3 Reducing Complexity

The first step is to reduce the complexity by transforming the three-dimensional problem into a two-dimensional one. This can be done by eliminating the distance component by transforming all objects onto a two-dimensional plane which represents the surface of the sphere with unity radius. The spheres are transformed into circles. Also the rotation in this plane is irrelevant as the circles are rotational invariant. So what we want to get are the two radii of the sun R and shadow caster r and the distance d between the centers of these two circles in the plane.

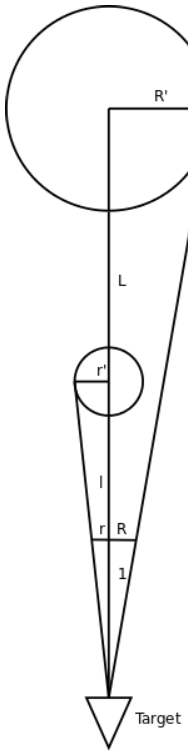

Figure 2: Transforming the spheres into circles using congruence of triangles. $R_{\odot} = R'$, $r_M = r'$

Figure 2 shows how the congruence of triangles can be used to transform the radii onto the plane. The lengths are given by

$$L = \left\| \vec{S} - \vec{P} \right\| \tag{1}$$

$$l = \left\| \vec{M} - \vec{P} \right\| \tag{2}$$

and thus the radii can be calculated as

$$R = \frac{R_\odot}{L} \tag{3}$$
$$r = \frac{r_M}{l}. \tag{4}$$

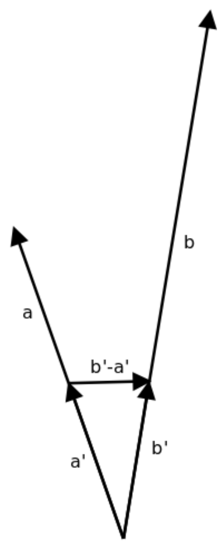

Figure 3: Calculating the distance projected onto the equidistant plane. $\vec{a} = \vec{S} - \vec{P}$, $\vec{b} = \vec{M} - \vec{P}$, $\vec{a'}$ and $\vec{b'}$ are the normalized vectors (having a length of one).

For the distance between the centers we simply calculate the length of the normalized vectors we already used for the length calculation as shown in figure 3 and equation 5.

$$d = \left\| \frac{\vec{S} - \vec{P}}{L} - \frac{\vec{M} - \vec{P}}{l} \right\| \tag{5}$$

1.4 Circle Intersection

What remains is a simple circle intersection problem as illustrated in figure 4.

The first thing to do is a distinction of cases:

1.4.1 No intersection

No intersection is given if the distance is bigger than the sum of the radii ($d \geqslant R + r$). In this case the illumination is 100 %, done.

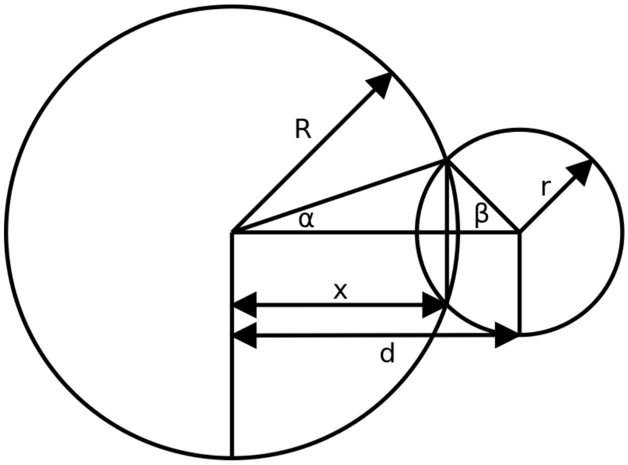

Figure 4: Intersection of two circles with radii R and r and with a distance d between the centers.

1.4.2 Umbra

If the radius r of the shadow caster is bigger than the distance d plus the radius R of the sun ($r \geqslant R + d$), then the shadow caster hides the sun completely and the intensity drops to zero for the given point $\vec{P}$.

1.4.3 Penumbra

This case can be split into two cases:

The shadow caster circle is completely inside the sun circle ($d + r \leqslant R$). In this case the amount of sun light visible is calculated via:

$$\frac{I}{I_0} = 1 - \frac{A_M}{A_S} = 1 - \frac{r^2\pi}{R^2\pi} = 1 - \left(\frac{r}{R}\right)^2 \tag{6}$$

The remaining case is an intersection of the two circles as seen in figure 4, for which more advanced calculations are necessary. First of all x has to be calculated. Using the circle equations

$$x^2 + y^2 = R^2 \tag{7}$$

$$(d - x)^2 + y^2 = r^2, \tag{8}$$

we get

$$x = \frac{R^2 + d^2 - r^2}{2d}. \tag{9}$$

Now there are at least two ways to calculate the actual area of the sun.

Way 1 The first way is using the equations

$$A_R = R^2 (\alpha - \sin(2\alpha)) \tag{10}$$
$$A_r = r^2 (\beta - \sin(2\beta)) \tag{11}$$

to calculate the area of a segment of the circle. α and β can be calculated via

$$\alpha = \cos^{-1}\left(\frac{x}{R}\right) \tag{12}$$
$$\beta = \cos^{-1}\left(\frac{d-x}{R}\right). \tag{13}$$

The intensity is then calculated as

$$\frac{I}{I_0} = 1 - \frac{A_R + A_r}{A_S}. \tag{14}$$

Way 2 The second way uses an integral to calculate the visible area of the sun as

$$A_v = 2 \cdot \int_{-R}^{x} \sqrt{R^2 - s^2}\ ds - 2 \cdot \int_{d-r}^{x} \sqrt{r^2 - (d-s)^2}\ ds \tag{15}$$

and then calculating the intensity as

$$\frac{I}{I_0} = \frac{A_v}{A_S}. \tag{16}$$